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A Congruence Relation for Wiener and Szeged Indices

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Abstract. In a recent paper [H. Lin, MATCH Communications in Mathematical and in Computer Chemistry 70 (2013) 575–582], a congruence relation for Wiener indices of a class of trees was reported. We now show that Lin's congruence is a special case of a much more general result.

1. Introduction

In this note we are concerned with simple graphs, without weighted or directed edges, and without self–loops. Let G be such graph. Let $V(G)$ and $E(G)$ be, respectively, the vertex and edge sets of G. The distance $d(u,v) = d(u,v|G)$ between the vertices u and v of G is the length of a shortest path connecting u and v. If G is connected, then

$$W(G) = \sum_{\{u,v\}\subseteq V(G)} d(u,v|G) \tag{1}$$

is referred to as the *Wiener index* of G. For details of the Wiener index see the survey [7] and the references cited therein.

In a recent paper, Lin [5] reported a congruence relation for the Wiener index of certain trees. In the terminology used in [5], a tree is said to have a path factor, if it has a spanning forest whose all components are paths of equal order. Let $\mathcal{T}(p,n)$ be the set of path–factor trees of order $p\,n$, having a spanning forest consisting of p paths of order n. Then Lin's congruence can be stated as:

Theorem 1.1. [5] *If $T_a, T_b \in \mathcal{T}(p,n)$, then $W(T_a) \equiv W(T_b)$ (mod n).*

In what follows we show that Theorem 1.1 is a special case of a much more general result. For this we first need to recall the definition of the *Szeged index* [2–4, 6].

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Let e be an edge of the graph G, connecting the vertices u and v. Denote by $n_1(e|G)$ the number of elements of the set $\mathcal{N}_1(e|G) = \{x \in V(G) \,|\, d(x,u) < d(x,v)\}$. Analogously, let $n_2(e|G)$ be the cardinality of the set $\mathcal{N}_2(e|G) = \{x \in V(G) \,|\, d(x,u) > d(x,v)\}$. Then the Szeged index is defined as

$$Sz(G) = \sum_{e \in E(G)} n_1(e|G)\, n_2(e|G) \,. \tag{2}$$

Although the right–hand sides of Eqs. (1) and (2) look quite dissimilar, the following result holds:

Theorem 1.2. [4] *If G is a connected graph, then the equality $Sz(G) = W(G)$ holds if and only if all blocks of G are complete graphs. In particular, the equality $Sz(G) = W(G)$ holds for trees.*

2. Generalizing Theorem 1.1

For $p \geq 2$, let $G_1, G_2, \ldots, G_p$ be connected graphs with disjoint vertex sets, each of order $n \geq 2$. Let Γ_0 be the (disconnected) graph of order $p\,n$, whose components are $G_1, G_2, \ldots, G_p$. Construct a graph Γ by adding $p-1$ new edges $e_1, e_2, \ldots, e_{p-1}$ to Γ_0, so that Γ becomes connected.

Evidently, $e_1, e_2, \ldots, e_{p-1}$ are cut-edges of Γ.

Theorem 2.1. *Let the graph Γ be constructed as described above. Then, irrespective of the actual position of the edges $e_1, e_2, \ldots, e_{p-1}$,*

$$Sz(\Gamma) \equiv \sum_{i=1}^{p} Sz(G_i) \ (mod\ n) \,.$$

Proof. Bearing in mind Eq. (2) and the structure of the graph Γ, we have

$$Sz(\Gamma) = \sum_{i=1}^{p} \sum_{e \in E(G_i)} n_1(e|\Gamma)\, n_2(e|\Gamma) + \sum_{k=1}^{p-1} n_1(e_k|\Gamma)\, n_2(e_k|\Gamma) \,. \tag{3}$$

Consider first the term $n_1(e|\Gamma)$ for some $e \in E(G_i)$. Let $j \neq i$.

In view of the way in which the graph Γ is constructed, if a vertex $w \in V(G_j)$ belongs to the set $\mathcal{N}_1(e|\Gamma)$, then (and only then) all vertices of G_j belong to $\mathcal{N}_1(e|\Gamma)$. Since all the subgraphs G_j, $j = 1, 2, \ldots, p$, are assumed to possess equal number of vertices (n), it follows that $n_1(e|\Gamma) = n_1(e|G_i) + \alpha\, n$ for some non-negative integer α.

By the same argument, $n_2(e|\Gamma) = n_2(e|G_i) + \beta\, n$ for some non-negative integer β.

Therefore,

$$n_1(e|\Gamma)\, n_2(e|\Gamma) \equiv n_1(e|G_i)\, n_2(e|G_i) \ (\text{mod}\ n)$$

and

$$\sum_{e \in E(G_i)} n_1(e|\Gamma)\, n_2(e|\Gamma) \equiv Sz(G_i) \ (\text{mod}\ n) \,. \tag{4}$$

By an analogous reasoning we conclude that for $k = 1, 2, \ldots, p-1$,

$$n_1(e_k|\Gamma) = \gamma\, n \qquad \text{and} \qquad n_2(e_k|\Gamma) = \delta\, n$$

where γ and δ are positive integers, such that $\gamma + \delta = p$. Consequently,

$$\sum_{k=1}^{p-1} n_1(e_k|\Gamma)\, n_2(e_k|\Gamma) \equiv 0 \ (\text{mod}\ n) \,. \tag{5}$$

Theorem 2.1 follows now by substituting (4) and (5) back into (3). □

3. Corollaries of Theorem 2.1

Corollary 3.1. *If $G_1 \cong G_2 \cong \cdots \cong G_p \cong G$, then, irrespective of the actual position of the edges $e_1, e_2, \ldots, e_{p-1}$,*

$$Sz(\Gamma) \equiv p\, Sz(G)\ (mod\ n)\ .$$

Bearing in mind Theorem 1.2, we arrive at:

Corollary 3.2. *If G_i , $i = 1, 2, \ldots, p$, are connected graphs, each of order n, whose all blocks are complete graphs (implying that also Γ has the same property), then*

$$W(\Gamma) \equiv \sum_{i=1}^{p} W(G_i)\ (mod\ n)\ . \tag{6}$$

In particular, relation (6) holds if Γ is a tree.

Corollary 3.3. *If, in addition to the conditions stated in Corollary 3.2, $G_1 \cong G_2 \cong \cdots \cong G_p \cong G$, then, irrespective of the actual position of the edges $e_1, e_2, \ldots, e_{p-1}$,*

$$W(\Gamma) \equiv p\, W(G)\ (mod\ n)\ . \tag{7}$$

In particular, relation (7) holds if Γ is a tree.

Let P_n denote the path of order n, and recall that its Wiener index is equal to $\binom{n+1}{3}$.

Corollary 3.4. *If $G_1 \cong G_2 \cong \cdots \cong G_p \cong P_n$, then irrespective of the actual position of the edges $e_1, e_2, \ldots, e_{p-1}$,*

$$W(\Gamma) \equiv p \binom{n+1}{3}\ (mod\ n)\ . \tag{8}$$

Lin's Theorem 1.1 is an immediate consequence of Corollary 3.4.

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